Numerical Solution of Burger's equation via Cole-Hopf transformed diffusion equation

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Abstract—A numerical method for solving Burger's equation via diffusion equation, which is obtained by using Cole-Hopf transformation, is presented. We compute the solution for transformed diffusion equation using explicit and implicit finite difference schemes and then use backward Cole-Hopf transformation to attain the solution for Burger's equation. This work also studies accuracy and numerical feature of convergence of the proposed method for specific initial and boundary values by estimating their relative errors.

Index Terms— Burger's equation, Cole-Hopf transformation, Diffusion equation, Discretization, Explicit scheme, Heat equation, Implicit scheme, Numerical solution, Neumann boundary condition

1 INTRODUCTION

THE one-dimensional Burger's equation has received an enormous amount of attention since the studies by J.M. Burger's in the 1940's, principally as a model problem of the interaction between nonlinear and dissipative phenomena.

The Burger's equation is nonlinear and one expects to find Phenomena similar to turbulence. However, as it has been shown by Hopf[2] and Cole[3], the homogeneous Burger's equation lacks the most important property attributed to turbulence: The solutions do not exhibit chaotic features like sensitivity with respect to initial conditions. This can explicitly shown using the Cole-Hopf transformation which transforms Burger's equation into a linear parabolic equation. From the numerical point of view, however, this is of importance since it allows one to compare numerically obtained solutions of the nonlinear equation with the exact one. This comparison is important to investigate the quality of the applied numerical schemes.

In this paper, we present the analytical solution of one-dimensional Burger's equation as an initial value problem in infinite spatial domain. Then we solve the diffusion equation, obtained from Burger's equation through Cole-Hopf transformation, using explicit and implicit finite difference schemes. Using solution data of the diffusion equation, we find solution for Burger's equation through backward Cole Hope transformation.

Then we find relative errors of the numerical methods to determine the accuracy of numerical methods.

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2 BURGER'S EQUATION AS AN IV PROBLEM

We consider the Burger's equation as an initial value problem[7]

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=\nu\frac{\partial^2 u}{\partial x^2} \tag{1}$$

$$\text{with I.C. } u(x,0)=u_0(x), \text{for} -\infty<x<\infty \tag{2}$$

3 ANALYTICAL SOLUTION OF BURGER'S EQUATION

After solving heat equation obtained from C-H(Cole Hopf) transformation and then using backward C-H transformation [3,4,9], we obtain following analytical solution of Burger's equation:

$$u(x,t)=\frac{\int_{-\infty}^{\infty}(x-y)\,exp\left[-\frac{(x-y)^2}{4\nu t}-\frac{1}{2\nu}\int_0^y u_0(z)dz\right]dy}{t\int_{-\infty}^{\infty}exp\left[-\frac{(x-y)^2}{4\nu t}-\frac{1}{2\nu}\int_0^y u_0(z)dz\right]dy} \tag{3}$$

4 Numerical evaluation of Analytical solution

We consider the bounded periodic function $u_0(x)=\sin x$ as initial condition and find the solution over the bounded spatial domain $[0,2\pi]$ at different time steps.

For the above initial condition we get the following analytical solution of Burger's equation,

$$u(x,t)=\frac{\int_{-\infty}^{\infty}(x-y)\,exp\left[-\frac{(x-y)^2}{4\nu t}+\frac{1}{2\nu}cos\,y\right]dy}{t\int_{-\infty}^{\infty}exp\left[-\frac{(x-y)^2}{4\nu t}+\frac{1}{2\nu}cos\,y\right]dy} \tag{4}$$

For very small ν, both numerator and denominator of (4) get more closed to zero or infinity which becomes very difficult to handle. So considering the value of ν arbitrarily very small, we cannot perform our numerical experiment.

We consider the value of ν as 0.1.

Again, for very small t, both numerator and denominator get much closed to zero and thus difficult to handle numerically.

We have found that for minimum value 0.1 of t the

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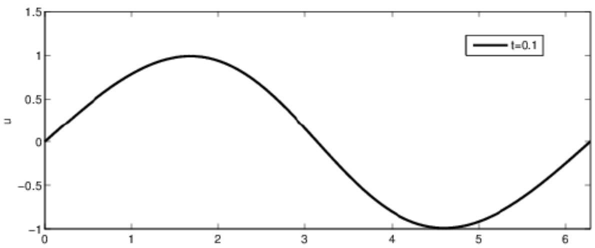
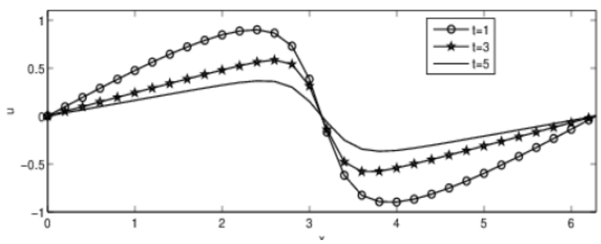
calculation is possible.



(a) Analytical solution of Burger's equation at $t = 0.1$



(b) Analytical solution of Burger's equation at $t = 1,3,5$

Fig. 1. Analytical solution of Burger's equation at different time steps.

5 Cole-Hopf Transformation

(1) can be linearized by Cole-Hopf transformation[3,4]

$$u(x,t) = -\frac{2\nu\phi_x}{\phi} \tag{5}$$

We perform the transformation in two steps.
First let us assume

$$u = \Psi_x$$

With this transformation (5) becomes,

$$\Psi_{xt} + \Psi_x\Psi_{xx} = \nu\Psi_{xxx}$$

$$\Rightarrow \Psi_{xt} + \frac{\partial}{\partial x}\left(\frac{1}{2}\Psi_x^2\right) = \nu\Psi_{xxx}$$

Which on integration w.r.to. x gives

$$\Psi_t + \frac{1}{2}\Psi_x^2 = \nu\Psi_{xx} \tag{6}$$

Then we make the transformation

$$\Psi = -2\nu \ln \phi$$

which turns (6) into

$$\phi_t = \nu\phi_{xx}$$

which is the well-known first order pde called heat or diffusion equation.

Solving (5) for ϕ, we have,

$$\phi(x,t) = Ce^{-\frac{1}{2\nu}\int u\,dx}$$

For $t = 0$,

$$\phi(x,0) = Ce^{-\frac{1}{2\nu}\int u_0\,dx}$$

From (5) it is clear that C has not effect on our final solution of Burger's equation.

So we can consider $\Phi(x,0)$ as

$$\phi(x,0) = e^{-\frac{1}{2\nu}\int_0^x u_0(z)dz} = \phi_0(x)(\text{let})$$

6 Cole-Hopf transformed diffusion equation

After Cole-Hopf transformation our problem turns into the following Cauchy problem for the Heat Equation.

$$\phi_t = \nu\phi_{xx} \tag{7}$$

$$\phi(x,0) = \phi_0(x) = e^{-\frac{1}{2\nu}\int_0^x u_0(z)dz} \tag{8}$$

For initial condition $u_0(x) = \sin x$, the initial condition of new problem is

$$\phi(x,0) = e^{-\frac{1}{2\nu}\int_0^x \sin z dz} = e^{\frac{\cos x}{2\nu}}$$

Now to obtain the transformed boundary condition, we consider the boundary condition of u as

$$u(0,t) = 0 = u(2\pi,t) \tag{9}$$

Using these boundary condition in

$$u(x,t) = -2\nu\frac{\phi_x}{\phi}$$

We obtain,

$$\frac{\phi_x(0,t)}{\phi} = 0 = \frac{\phi_x(2\pi,t)}{\phi} \tag{10}$$

Solving (5) for ϕ, we have,

$\phi(x,t) = Ce^{-\frac{1}{2\nu}\int u dx}$, where C is integrating constant which guarantees us that ϕ does not vanish for every choices of x and t unless we choose C as zero (which we should not choose because for $C = 0$, singularity occurs in calculating the value of u at each point)

Since ϕ does not vanish for every choices of x and , so from (10), we have

$$\phi_x(0,t) = 0 = \phi_x(2\pi,t)$$

So finally we have obtained the following problem concerning diffusion equation as an Initial-Boundary value problem with Neumann boundary conditions.

$$\phi_t = \nu\phi_{xx} \tag{11}$$

$$\begin{aligned} I.C.\ \phi(x,0) &= e^{\frac{\cos x}{2\nu}} \\ B.C.\ \phi_x(0,t) &= 0 = \phi_x(2\pi) \end{aligned} \tag{12}$$

7 An explicit scheme to solve Diffusion equation with Neumann boundary conditions:

To find an explicit scheme, we discretize the $x - t$ plane by choosing a mesh width $h \equiv \Delta x$ and a time step $k \equiv \Delta t$, and define the discrete mesh points (x_i, t_n) by

$$x_i = a + ih, i = 0,1, \dots\dots\dots M$$
$$and\ t_n = nk, n = 0,1, \dots\dots\dots N$$

Where,

$$M = \frac{b-a}{h}\ and\ N = \frac{T}{k}$$

We discretize $\frac{\partial\phi}{\partial t}$ and $\frac{\partial^2\phi}{\partial x^2}$ at any discrete point (x_i, t_n) as follows:

$$\frac{\partial\phi}{\partial t} \approx \frac{\phi_i^{n+1}-\phi_i^n}{k} \tag{13}$$

$$\frac{\partial^2\Phi}{\partial x^2} = \frac{\phi_{i+1}^n - 2\phi_i^n + \phi_{i-1}^n}{h^2} \tag{14}$$

Inserting (13) and (14) in (11), our discrete version of diffusion equation formulates as the second order finite difference scheme of the form:

$$\frac{\phi_i^{n+1}-\phi_i^n}{k} = \nu\frac{\phi_{i+1}^n - 2\phi_i^n + \phi_{i-1}^n}{h^2} \tag{15}$$

where ϕ_i^n is the value of ϕ at the point (x_i, t_n) in $x - t$ plane.

(15) can be rewritten as

$$\phi_i^{n+1} = (1-2\gamma)\phi_i^n + \gamma(\phi_{i+1}^n + \phi_{i-1}^n) \tag{16}$$

where, $\gamma = \frac{\nu k}{h^2}$

Since we have given Neumann boundary conditions, so we are not been able to find boundary values directly.

But the derivatives of ϕ w.r.t. x vanishes at boundary points for each time step t_n.

So we can consider that the values of ϕ at two points (one is boundary and the other is the nearest point of boundary according to our discretisation) for each boundary are same at each time step which comes from the following fact:

$$\frac{\phi_1^n - \phi_0^n}{h} = \frac{\phi_M^n - \phi_{M-1}^n}{h} = 0$$

$$\Rightarrow \phi_1^n = \phi_0^n \ and\ \phi_M^n = \phi_{M-1}^n \tag{17}$$

Now we have given initial values ϕ_i^0.

To calculate the values $\phi_1^1, \phi_2^1, \phi_3^1, \dots\dots\dots\dots.., \phi_{M-1}^1$, we put $n = 0$, in (16) and get

$$\phi_i^1 = (1-2\gamma)\phi_i^0 + \gamma(\phi_{i+1}^0 + \phi_{i-1}^0) \tag{18}$$

Inserting $i = 1,2,3, \dots\dots.., M-1$ in (18), we get the values $\phi_1^1, \phi_2^1, \phi_3^1, \dots\dots\dots, \phi_{M-1}^1$ and to obtain ϕ_0^1 and ϕ_M^1 , we replace n by 1 in (17) and find

$$\phi_1^1 = \phi_0^1 \ and\ \phi_M^1 = \phi_{M-1}^1$$

After calculating the values of ϕ at t_1, we can find the values of ϕ at t_2 using the same process.

Now let values of ϕ at all discretized points have been calculated for $t = t_n$.

Then using (16), we can calculate $\phi_1^{n+1}, \phi_2^{n+1}, \phi_3^{n+1}, \dots\dots\dots\dots, \phi_{M-1}^{n+1}$ and we use (17) to calculate the boundary values ϕ_0^{n+1} and ϕ_M^{n+1}.

Proceeding in this way, we finally obtain the values of ϕ at each of our discretized point.

Choosing $\nu = 0.1$, $h = 0.1$ and $k = 0.01$, we have performed the explicit scheme for $t = 0$ to 5

8 An implicit scheme to solve heat equation with Neumann boundary condition

To obtain an implicit scheme, we discretize $\frac{\partial\phi}{\partial t}$ and $\frac{\partial^2\phi}{\partial x^2}$ at any discrete point (x_i, t_{n+1}) as follows:-

$$\frac{\partial\phi}{\partial t} \approx \frac{\phi_i^{n+1}-\phi_i^n}{k} \tag{19}$$

$$\frac{\partial^2\phi}{\partial x^2} = \frac{\phi_{i+1}^{n+1} - 2\phi_i^{n+1} + \phi_{i-1}^{n+1}}{h^2} \tag{20}$$

Inserting (19) and (20) in (11), the discrete version of our diffusion equation formulates as the second order finite difference scheme of the form

$$\frac{\phi_i^{n+1}-\phi_i^n}{k} = \nu\frac{\phi_{i+1}^{n+1} - 2\phi_i^{n+1} + \phi_{i-1}^{n+1}}{h^2} \tag{21}$$

Where ϕ_i^n is the value of ϕ at the point (x_i, t_n) in $x - t$ plane.

(21) can be rewritten as

$$(1+2\gamma)\phi_i^{n+1} - \gamma(\phi_{i+1}^{n+1} + \phi_{i-1}^{n+1}) = \phi_i^n \tag{22}$$

Where, $\gamma = \frac{\nu k}{h^2}$

Since we have given Neumann boundary conditions, so we have not been able to find boundary values directly as discussed in the previous section. From (17), we have the following relations

$$\phi_1^n = \phi_0^n \ and\ \phi_M^n = \phi_{M-1}^n \tag{23}$$

Now we have given the initial values ϕ_i^0.

To calculate the values of $\phi_1^1, \phi_2^1, \phi_3^1, \dots\dots\dots\dots.., \phi_{M-1}^1$. We put n = 0 in (22) and get

$$(1+2\gamma)\phi_i^1 - \gamma(\phi_{i+1}^1 + \phi_{i-1}^1) = \phi_i^0 \tag{24}$$

Inserting i = 1,2,3, ……….., M − 1 in (24) and using (23), we get the following linear system of $M-1$ equations in $M-1$ variables-

$$\left.\begin{array}{c}(1+\gamma)\phi_1^1 - \gamma\phi_2^1 = \phi_1^0 \\ -\gamma\phi_1^1 + (1+2\gamma)\phi_2^1 - \gamma\phi_3^1 = \phi_2^0 \\ -\gamma\phi_2^1 + (1+2\gamma)\phi_3^1 - \gamma\phi_4^1 = \phi_3^0 \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ -\gamma\phi_{M-3}^1 + (1+2\gamma)\phi_{M-2}^1 - \gamma\phi_{M-1}^1 = \phi_{M-2}^0 \\ -\gamma\phi_{M-2}^1 + (1+\gamma)\phi_{M-1}^1 = \phi_{M-1}^0\end{array}\right\} \tag{25}$$

The above system can be written as

$$Ax = b \tag{26}$$

Where,

$$A = \begin{pmatrix} 1+\gamma & -\gamma & 0 & \cdots & \dots & 0 \\ -\gamma & 1+2\gamma & -\gamma & \ddots & & \vdots \\ 0 & -\gamma & 1+2\gamma & -\gamma & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & -\gamma & 1+2\gamma & -\gamma \\ 0 & \dots & \dots & 0 & -\gamma & 1+\gamma \end{pmatrix},$$

$$X = (\phi_1^1, \phi_2^1, \phi_3^1, \dots\dots\dots.., \phi_{M-2}^1, \phi_{M-1}^1)^T$$

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and

$$b = (\phi_1^0, \phi_2^0, \phi_3^0, \dots \dots, \phi_{M-2}^0, \phi_{M-1}^0)^T$$

Solving (26), we get

$X = (\phi_1^1 \quad \phi_2^1 \quad \phi_3^1 \quad \dots \quad \dots \quad \phi_{M-2}^1 \quad \phi_{M-1}^1)$ and so the values $\phi_1^1, \phi_2^1, \phi_3^1, \dots \dots \dots \dots, \phi_{M-1}^1$ and to obtain ϕ_0^1 and ϕ_M^1, we replace n by 1 in (23) and find

$$\phi_1^1 = \phi_0^1 \ and \ \phi_M^1 = \phi_{M-1}^1$$

So we have been able to calculate values of ϕ at all discretized points for $n = 1$.

After calculating the values of ϕ at t_1, one can find the values of ϕ at t_2 using the same process.

Proceeding in this way, we finally obtain the values of ϕ at each of our discretized point.

Choosing $\nu = 0.1$, $h = 0.1$ and $k = 0.01$, we have performed the implicit scheme for t = 0 to 5.

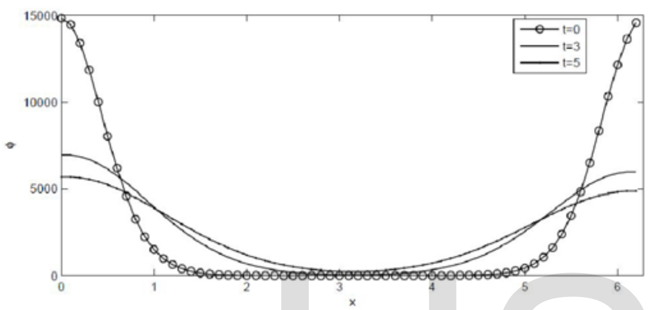

Fig. 2. Numerical solution of diffusion equation using an explicit/implicit scheme with $h = 0.1, k = 0.01$ and $\nu = 0.1$ at different times

9 Calculation of derivatives of ϕ w.r.t. x at different discretized points

Since the derivatives of ϕ have to be taken w.r.t.x , so we just consider the values of ϕ at a fixed time and then calculate the values of ϕ_x from that data.

At any discretized time $t = t_n$, the values ϕ_i^n are known.

Let $D\phi_i^n$ denote the derivative of ϕ at (x_i, t_n).

Then $D\phi_i^n$ can be calculated from the first order centered difference formula:-

$$\frac{\partial \phi}{\partial x} \approx \frac{\phi_{i+1}^n - \phi_{i-1}^n}{2h} \tag{27}$$

So we define

$$D\phi_i^n = \frac{\phi_{i+1}^n - \phi_{i-1}^n}{2h} \tag{28}$$

The derivatives $D\phi_0^n$ and $D\phi_M^n$ at the end points are known.

The other derivatives $D\phi_1^n, D\phi_2^n, D\phi_3^n, \dots \dots \dots \dots \dots, D\phi_{M-1}^n$ can be calculated by putting $i = 1,2,3, \dots \dots \dots, M-1$ in (28).

For $\nu = 0.1, h = 0.1, k = 0.01$ the values of ϕ_x are pictorized in figure.

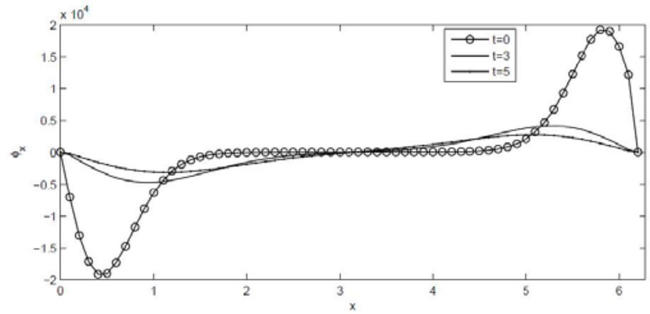

Fig. 3. Numerical calculation of ϕ_x for $h = 0.1, k = 0.01, \nu = 0.1$

10 Calculating the required solution

Once the values of ϕ and ϕ_x are known at all discrete points, then the values of u at discrete points can be calculated from the following discrete version of (5).

$$u_i^n = -2\nu \frac{D\phi_i^n}{\phi_i^n} \tag{29}$$

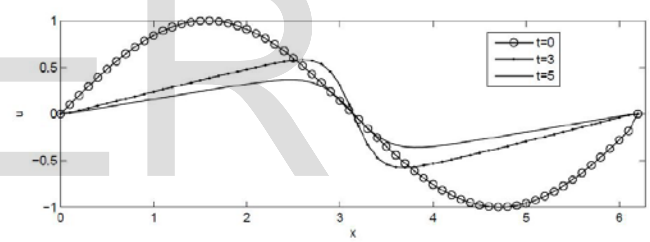

Fig. 4. Solution of Burger's equation

11 Relative error

We compute the relative error in $L_1 - norm$ defined by

$$||e||_1 := \frac{||u_e - u_n||_1}{||u_e||_1}$$

for all time $t = 0$ to $t = 5$, where u_e is the exact solution and u_n is the numerical solution computed by our proposed method.

After computation of relative errors, we show the convergence of each scheme by plotting relative errors for different pairs of (h, k).

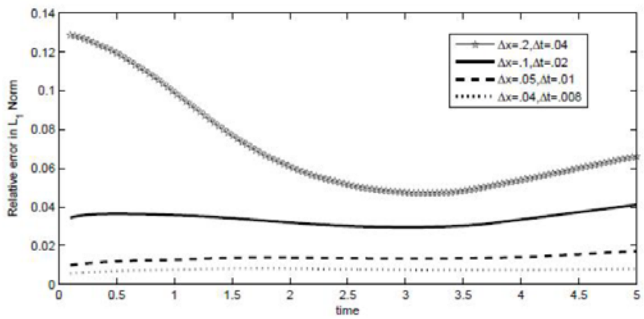

Fig. 5. Convergence of relative errors to zero when we use explicit scheme in solving heat equation

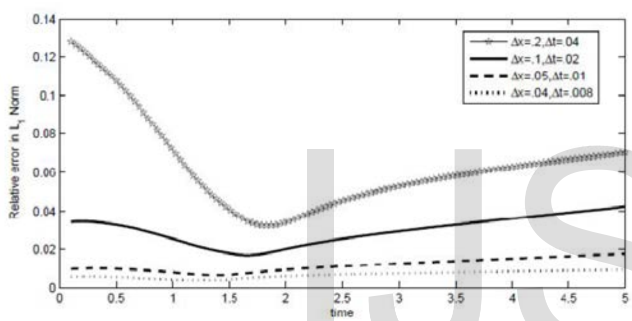

Fig. 6. Convergence of relative errors to zero when we use implicit scheme in solving heat equation

12 Conclusion

When we solve Burger's equation by applying an explicit scheme directly, then the stability condition also depends on value of ν and if we use implicit scheme directly then numerical stability reduces as ν decreases[11].

Due to these drawbacks of direct use of explicit and implicit scheme on non-linear burger's equation[11], we first transformed non-linear burger's equation to linear heat/diffusion equation and applied explicit/implicit scheme on that diffusion equation. When we solve heat equation using explicit scheme, stability condition doesn't depend on ν, so we can consider also small values of ν and also if we solve heat equation using implicit scheme, then we don't need to consider any stability condition.

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